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AND DIFFUSE REFLECTION FROM CYLINDRICAL  
REGIONS

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ON THE PRINCIPLE OF INVARIANT IMBEDDING  
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By

Richard Bellman and Robert Kalaba

1. Introduction.

The importance of various principles of invariance in treating problems of radiative transfer has been discussed by Ambarzumian<sup>1,2</sup> and Chandrasekhar<sup>4</sup>, who applied these principles to a variety of problems connected with plane-parallel regions. However, the application of these concepts to the corresponding problems for cylindrical and spherical regions is not immediate. Chandrasekhar<sup>5</sup> has observed: "The extension of the approximate methods of solution to other geometries is straightforward; but it is not equally apparent what the general principles are which will play the same unique role as the principle of invariance in plane-parallel atmospheres."

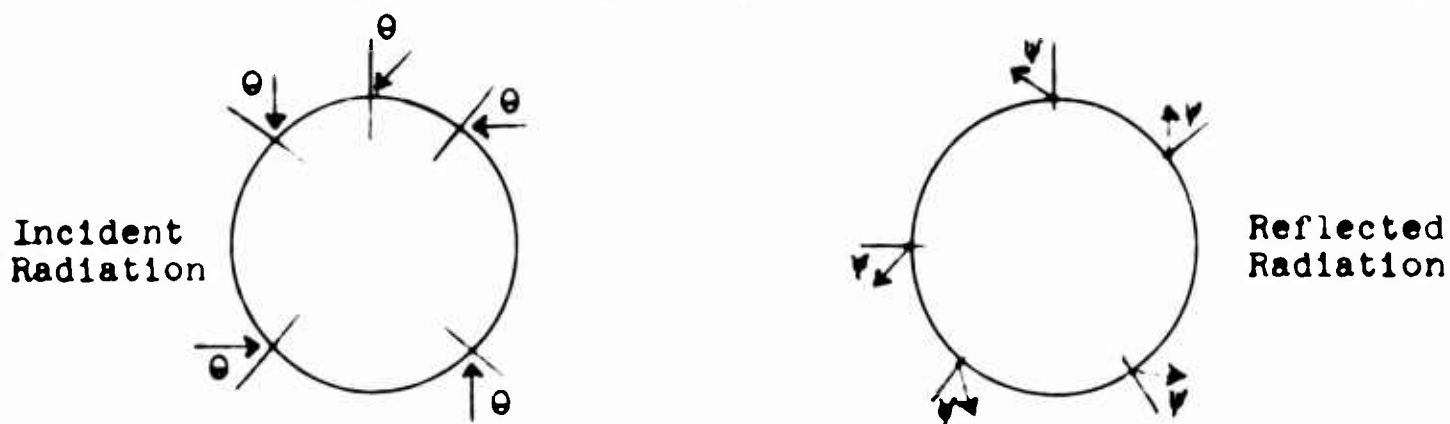
In our first paper<sup>3</sup>, we introduced the principle of invariant imbedding as an extension of the ideas of Ambarzumian and Chandrasekhar, and utilized this principle to treat the case of inhomogeneous plane-parallel regions. In this paper, we propose to show that this new concept enables us to treat cylindrical, and analogously spherical, geometries in a unitary fashion, and thus provides an answer to the query of Chandrasekhar.

Analytic and computational questions arising from this treatment will be discussed in greater detail in a subsequent publication.

## 2. The Physical Model.

Let us now consider the problem of determining the intensity of diffusely reflected radiation from the surface of a circular cylinder. We assume that the symmetry properties of the medium are such that it suffices to consider interactions in a cross-section perpendicular to the axis, and that the absorption and scattering parameters of elemental volumes depend only upon distance from the axis, and, finally, that the scattering is isotropic.

Consider an incoming flux of radiation making a constant angle  $\theta$  with the radius of the circle, as indicated in the figure below, and taken to have intensity one per unit length normal to the direction of propagation. This is the appropriate definition of flux for circular regions corresponding to the usual one for plane regions.



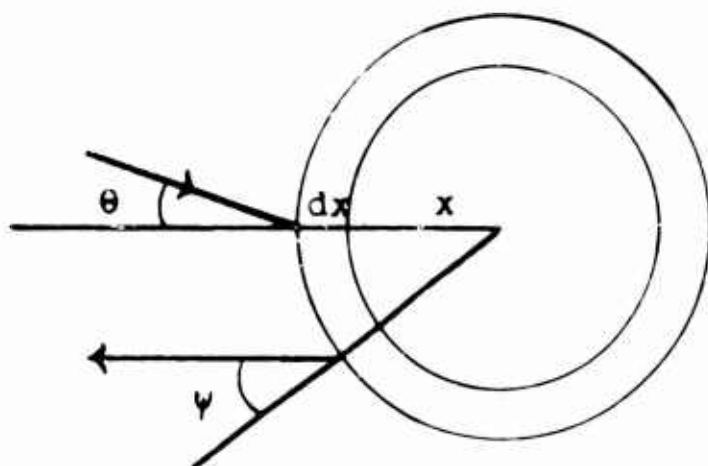
As in the plane case, the ratio of scattering coefficient to absorption coefficient is denoted by  $\lambda(x)/(1-\lambda(x))$ , where  $x$  is the distance from the center of the circle.

We now consider the intensity of flux of reflected radiation making a fixed angle  $\psi$  with the extended radius of a cylinder of radius  $x$ , due to an incoming flux at angle  $\theta$ , as above. This

intensity is considered to be a function  $r(v, u, x)$  of  $u = \cos \theta$ ,  $v = \cos \psi$ , and  $x$ . In this way, we imbed the particular process within a continuous family of processes.

### 3. Application of the Principle of Invariant Imbedding.

In order to obtain a functional equation for  $r(v, u, x)$ , we consider the scattering from a cylinder of radius  $x + dx$  as the result of scattering from the infinitesimal annulus,  $(x, x+dx)$ , and the cylinder of radius  $x$ .



Although we are definitely considering multiple scattering, our functional equations are considerably simplified by the fact that, to higher order terms, only one scattering can occur within the annulus.

Proceeding as in Reference 3 which in turn follows the procedures of Ambarzumian and Chandrasekhar, we obtain the following nonlinear integro-differential equation

$$r_x + \left(\frac{1}{u} + \frac{1}{v}\right)r + \left(\frac{1-v^2}{xv}\right)r_v + \left(\frac{1-u^2}{xu}\right)r_u = \frac{\lambda(x)}{4v} + \frac{\lambda(x)}{2v} \int_0^1 r(w, u, x) dw$$

$$(1)$$

$$+ \frac{\lambda(x)}{2} \int_0^1 r(v, z, x) \frac{dz}{z} + \lambda(x) \int_0^1 r(w, u, x) dw \int_0^1 r(v, x, z) \frac{dz}{z},$$

together with the boundary condition,  $r(v, u, 0) = 0$ .

This differs from the corresponding equation for the inhomogeneous parallel-plane case, derived in our first paper<sup>3</sup>, by the presence of the partial derivatives with respect to  $u$  and  $v$ . These terms arise from the fact that the radiation incident at angle  $\theta$  on a circle of radius  $x + dx$  is incident, to first order terms, at an angle  $x + (\tan \theta) dx/x$  on a circle of radius  $x \rightarrow \theta$ .

#### 4. Reduction in Dimensionality.

As the equation stands, it is an equation for a function of three variables,  $r(v,u,x)$ , two space-like variables,  $u$  and  $v$ , and one time-like variable,  $x$ . Let us now show that this equation may be replaced by two equations for the functions  $P(v,x)$  and  $Q(u,x)$  defined as

$$\begin{aligned} P(v,x) &= \int_0^1 r(v,z,x) dz/z, \\ Q(u,x) &= \int_0^1 r(w,u,x) dw. \end{aligned} \tag{1}$$

The equation in (3.1) has the form

$$L(r) = F(P(v,x), Q(v,x)), \tag{2}$$

where  $L$  is a linear partial differential operator. Solving for  $r$  in terms of  $F$ , using characteristic theory or otherwise, we obtain a relation of the form

$$r = \bigwedge(F) \tag{3}$$

where  $\wedge$  is a linear operator. Using this relation, we form  $P(v,x)$  and  $Q(u,x)$  as defined above.

$$\begin{aligned} P(v,x) &= \int_0^1 \wedge(F) dz/z, \\ Q(u,x) &= \int_0^1 \wedge(F') dw. \end{aligned} \tag{4}$$

This last set of equations constitute a system of two nonlinear integral equations for the functions  $P(v,x)$  and  $Q(u,x)$ .

Computation and analytic aspects of these equations will be presented subsequently.

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4. S. Chandrasekhar, Radiative Transfer, Oxford (1950).
5. See p. 364 of Reference 4, above.